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THE MOTION OF WHEELED ROBOTS†

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The equations of motion of three-wheeled robots with two drive wheels and one passive caster wheel are derived and investigated. The control of longitudinal motion and turns of such a robot is implemented by appropriate control of the independent motors of the drive wheels. The research is carried out under the assumption that the robot is moving on a horizontal plane surface and that the wheels do not slip. A system of two non-linear equations with two controls is obtained for the non-holonomic system considered. The dependence of the phase portrait on the values of the constant controls and parameters of the system, taking into account the asymmetry of the robot, is investigated. The results obtained are not only of theoretical but also of practical interest.

1. THE EQUATIONS OF MOTIONS

We will assume that the robot is moving on a plane horizontal surface – a polygon. We will introduce an absolute system of coordinates $C\xi_1\xi_2\xi_3$ connected with the polygon, the unit vectors ξ_1 and ξ_2 are in the horizontal plane, and ξ_3 is directed along the vertical and forms a right triple with them (Fig. 1). We will connect the system of coordinates $Ox_1x_2x_3$ with the body of the robot in such a way that the point O is situated on the axis of the drive wheels, the unit vector $\mathbf{X}_3$ is parallel to ξ_3, the unit vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ form a right triple with it and the unit vector $\mathbf{X}_1$ is directed forward along the body. The vector $\mathbf{a} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2$ specifies the position of the centre of gravity of the body, b is half the distance between the wheels and φ is the course angle of the robot. We will consider the motion of the robot using the coupled axes $Ox_1x_2x_3$ and adopt the hypothesis of non-sliding wheels. Then the absolute velocity of the point O and the absolute angular velocity of the robot will be written in the form

$$\mathbf{V} = V\mathbf{X}_1, \quad \boldsymbol{\omega} = \omega\mathbf{X}_3$$

The fact that the vector of the linear velocity is directed along the x_1 axis represents a non-integrable constraint on the velocity, which corresponds to the definition of a non-holonomic constraint.

The robot considered, course a third passive caster wheel but, using the fact that usually its influence on the character of the motion is small; we will replace it with a slidable strut for simplification.

We will determine the absolute linear velocities $\mathbf{V}_m$, $\mathbf{V}_+$ and $\mathbf{V}_-$ of the centre of gravity of the body and the centres of the right and left drive wheels.

$$\mathbf{V}_m = (V - a_2\omega)\mathbf{X}_1 + a_1\omega\mathbf{X}_2, \quad \mathbf{V}_\pm = (V \pm b\omega)\mathbf{X}_1$$

If $\omega_\pm$ are the axial angular velocities of the wheels and r is the radius of the wheel, then for the non-sliding wheels

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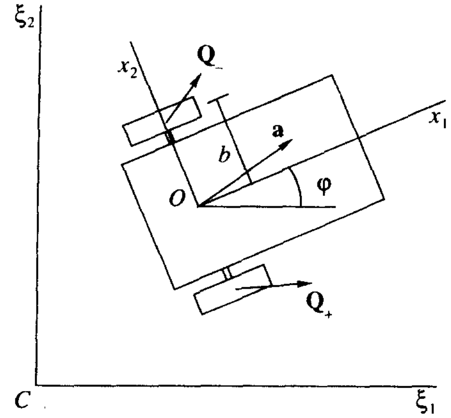


Fig. 1

$$\omega_\pm = V_\pm/r = (V \pm b\omega)/r \tag{1.1}$$

We will introduce the mass m_0 of the body, the mass m_w of the wheel, the central moment of inertia of the body J_0 about the vertical axis and the central axial J_w' and equatorial J_w'' moments of inertia of the wheel. The moments and singular moments about centres of mass of the body and wheels have the form

$$\begin{aligned} \mathbf{P}_0 &= m_0[(V - a_2\omega)\mathbf{X}_1 + a_1\omega\mathbf{X}_2], \quad \mathbf{K}_0 = J_0\omega\mathbf{X}_3 \\ \mathbf{P}_\pm &= m_w(V \pm b\omega)\mathbf{X}_1, \quad \mathbf{K}_\pm = J_w'\omega_\pm\mathbf{X}_2 + J_w''\omega\mathbf{X}_3 \end{aligned} \tag{1.2}$$

Note that for the plane horizontal motion assumed, the gravity forces of the elements of the robot will only appear in the expressions for the vertical components of the forces, which will later be eliminated from the equations and hence can be ignored from the beginning.

We will denote by

$$\mathbf{S}_\pm = -a_1\mathbf{X}_1 - (a_2 \pm b)\mathbf{X}_2 \tag{1.3}$$

the radius vectors of the centres of the wheels relative to the centre of masses of the body, $\mathbf{F}_\pm$ and $\mathbf{M}_\pm$ are the forces and moments acting on the wheels from the side of the body (the reference points are the centres of the wheels), and $\mathbf{Q}_\pm$ are the forces acting on the wheels from the side of the polygon at the contact points.

We can now write the equations of motion for each of the three rigid bodies considered by applying the fundamental theorems of mechanics

$$\begin{aligned} \frac{d\mathbf{P}_0}{dt} &= -\mathbf{F}_+ - \mathbf{F}_-, \quad \frac{d\mathbf{K}_0}{dt} = -\mathbf{S}_+ \times \mathbf{F}_+ - \mathbf{S}_- \times \mathbf{F}_- - \mathbf{M}_+ - \mathbf{M}_- \\ \frac{d\mathbf{P}_\pm}{dt} &= \mathbf{Q}_\pm + \mathbf{F}_\pm, \quad \frac{d\mathbf{K}_\pm}{dt} = \mathbf{M}_\pm - r\mathbf{X}_3 \times \mathbf{Q}_\pm \end{aligned} \tag{1.4}$$

Eliminating $\mathbf{F}_\pm$ and $\mathbf{M}_\pm$ from the first two equations and putting

$$\mathbf{P} = \mathbf{P}_0 + \mathbf{P}_+ + \mathbf{P}_- = [(m_0 + 2m_w)V - m_0a_2\omega]\mathbf{X}_1 + m_0a_1\omega\mathbf{X}_2$$

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_+ + \mathbf{K}_- = \frac{2}{r}J_w'V\mathbf{X}_2 + (J_0 + 2J_w'')\omega\mathbf{X}_3$$

we obtain two vector equations

$$\begin{aligned} &\frac{d\mathbf{P}}{dt} = \mathbf{Q}_+ + \mathbf{Q}_- \\ &\frac{d\mathbf{K}}{dt} + \mathbf{S}_+ \times \frac{d\mathbf{P}_+}{dt} + \mathbf{S}_- \times \frac{d\mathbf{P}_-}{dt} = (\mathbf{S}_+ - r\mathbf{X}_3) \times \mathbf{Q}_+ + (\mathbf{S}_- - r\mathbf{X}_3) \times \mathbf{Q}_- \end{aligned} \tag{1.5}$$

Now, taking into account the rule of differentiation along moving axes and Eqs (1.1)–(1.3) and denoting the time derivative by a dot, we will write the first equation of (1.5) in projections on to the x_1, x_2 axes, the second in a projection on to the x_3 axis and the last equation of (1.4) for both wheels will be written in a projection on to the x_2 axis

$$(m_0 + 2m_w)V^{\cdot} - m_0 a_2 \omega^{\cdot} - m_0 a_1 \omega^2 = Q_{1+} + Q_{1-}$$

$$m_0 a_1 \omega^{\cdot} + (m_0 + 2m_w)V\omega - m_0 a_2 \omega^2 = Q_{2+} + Q_{2-}$$

$$(J_0 + 2J''_w + 2m_w b^2)\omega^{\cdot} + 2m_w a_2 V^{\cdot} - 2m_w a_1 V\omega =$$
$$= -a_1(Q_{2+} + Q_{2-}) + a_2(Q_{1+} + Q_{1-}) + b(Q_{1+} - Q_{1-})$$

$$r^{-1}J'_w(V^{\cdot} \pm b\omega^{\cdot}) = M_{\pm} - rQ_{1\pm}$$

The equations contain two axial control torques $M_{\pm} = M_{2\pm}$ applied to the wheels from the side of the body. Eliminating the combination $Q_{2+} + Q_{2-}$ and the reactions Q_{1+} and Q_{1-} in an obvious was we arrive at the following system of two non-linear dynamic equations

$$\left(m_0 + 2m_w + \frac{2}{r^2}J'_w\right)V^{\cdot} - m_0 a_2 \omega^{\cdot} - m_0 a_1 \omega^2 = \frac{1}{r}(M_+ + M_-)$$

$$\left(J_0 + 2J''_w + 2m_w b^2 + m_0 a_1^2 + \frac{2}{r^2}J'_w b^2\right)\omega^{\cdot} + 2\left(m_w + \frac{J'_w}{r^2}\right)a_2 V^{\cdot} + \tag{1.6}$$

$$+ m_0 a_1 V\omega - m_0 a_1 a_2 \omega^2 = \frac{b}{r}(M_+ - M_-) + \frac{a_2}{r}(M_+ + M_-)$$

Suppose the control torques M_+ and M_- are formulated as follows, which is a standard way when describing drives with direct-current motors:

$$M_{\pm} = c_{1\pm}u_{\pm} - c_{2\pm}\omega_{\pm}$$

where $c_{1\pm}$ and $c_{2\pm}$ are the parameters of the various drives and $u_{\pm}$ are the control voltages.

Taking Eqs (1.1) into the account we have

$$M_+ \pm M_- = c_{1\pm}u_+ \pm c_{1-}u_- - (c_{1+} \pm c_{2-})\frac{V}{r} - (c_{2+} \mp c_{2-})\frac{b\omega}{r} \tag{1.7}$$

We will introduce a new variable instead of V and also new parameters

$$\upsilon = \frac{V}{b}; \quad \theta = \frac{2}{m_0}\left(m_w + \frac{1}{r^2}J'_w\right), \quad m = m_0(1+\theta), \quad a = \frac{m_0 a_1}{mb}, \quad e = \frac{m_0 a_2}{mb}$$

$$J = \frac{1}{mb^2}[J_0 + 2J''_w + b^2 m_0 \theta + m_0 a_1^2], \quad \sigma = \frac{c_{2+} + c_{2-}}{mr^2}, \quad \gamma = \frac{c_{2+} - c_{2-}}{c_{2+} + c_{2-}} \tag{1.8}$$

$$p = \frac{1}{mrb}(c_{1+}u_+ + c_{1-}u_-), \quad q = \frac{1}{mrb}(c_{1+}u_+ - c_{1-}u_-)$$

Equations (1.6), with the new notation (1.7), will become

$$\begin{aligned} &\upsilon^{\cdot} - e\omega^{\cdot} - a\omega^2 + \sigma\upsilon + \gamma\sigma\omega = p \\ &(J + \theta e^2)\omega^{\cdot} + a\upsilon\omega - ea\omega^2 + \sigma\omega + \gamma\sigma\upsilon + e\sigma\upsilon + e\gamma\sigma\omega = q + ep \end{aligned} \tag{1.9}$$

The parameters $J > 0$ and $\theta > 0$ are determined by the inertial-mass characteristics of the system, a, e give the position of the centre of gravity of the body relative to the wheels, $\sigma \geqslant 0$ is the normalized viscous friction in the wheel axes, $\gamma(|\gamma| < 1)$ is a parameter defining the asymmetry of the friction, and p and q are the controls of the longitudinal velocity and the rotation of the body. Below are will assume that $a > 0$, since the reverse situation is equivalent to a change in the sign of the velocity.

We will examine the behaviour of the system in the case when the control signals p and q are constant. Then system (1.9) is autonomous and the use of the phase plane ω, υ is very convenient for investigating the motions for different values of the parameters.

2. SIMPLE CASES

For special values of the parameters, Eqs (1.9) transform into the equations of non-holonomic systems, considered in the classical publications [1–4]. We will cite these results here.

Chaplygin's sledge. When $p = q = e = \sigma = \gamma = 0$ system (1.9) takes the form

$$\upsilon^{\boldsymbol{\cdot}} - a\omega^2 = 0, \quad J\omega^{\boldsymbol{\cdot}} + a\upsilon\omega = 0 \tag{2.1}$$

and is identical with the system obtained and analysed by Chaplygin [1] and Caratheodory [3], who examined the inertial motion along the horizontal plane of a "Chaplygin's sledge", a non-holonomic mechanical system, representing a rigid body resting on the plane with two sliding points and a point of the skate blade. The position of the contact point, of the skate coincides here with the centre of the section connecting the fixing points of the wheels of the mobile robot.

The phase plane for the case considered is shown in Fig. 2. The stationary points $\omega_0 = 0$, $\upsilon_0 = \text{const}$ of the system (2.1) fill the whole ordinate axis. It is obvious that the motions of the robot with constant velocity along a straight lines coincide with these stationary points. System (2.1) has the integral

$$\upsilon^2 + J\omega^2 = \text{const}$$

which defines a family of ellipses ($J > 0$) – phase trajectories on the ω, υ plane. For $a > 0$ the representative point moves along the phase trajectory from below upwards; consequently, the stationary points $\upsilon_0 < 0$ are unstable and $\upsilon_0 > 0$ are stable. Hence the motions of a robot with the centre of gravity of the body behind the wheels are unstable, and those with the centre of gravity of the body in front of the wheels are stable. The cuspidal point of the trajectory of motion of the robot coincides with the intersection of the phase trajectory with the abscissa axia.

Appell's mechanism. When $q = e = \sigma = \gamma = 0$ system (1.9) takes the form

$$\upsilon^{\boldsymbol{\cdot}} - a\omega^2 = p, \quad J\omega^{\boldsymbol{\cdot}} + a\upsilon\omega = 0 \tag{2.2}$$

these equations are identical with the equations obtained and analysed by Appell [2] and later by Hamel [4] for a non-holonomic mechanical system, which differs from "Chaplygin's sledge" in the fact that, instead of a skate, it has a wheel, on which a constant torque acts, produced by means of a load on a thread positioned over a fixed block on the top of the body and wound round a pulley coaxial with the wheel.

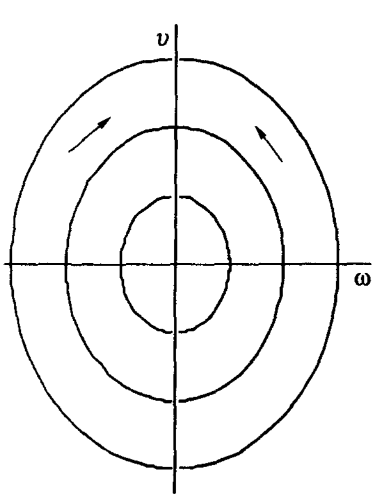


Fig. 2

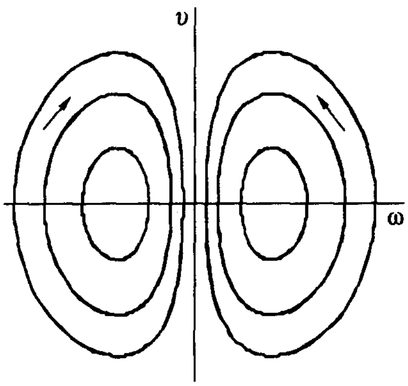


Fig. 3

We will present some results from these publications. If $p > 0$, there are no stationary points; if $p < 0$, we have have two stationary points

$$v_0 = 0, \quad \omega_0 = \pm\sqrt{-p/a} \tag{2.3}$$

The autonomous equations (2.2), eliminating the time, can be reduced to the form

$$(p/\omega^2 + a)d(\omega^2) = -(a/J)d(v^2) \tag{2.4}$$

Equation (2.4) obviously has the integral

$$a\omega^2 + p\ln(\omega^2) + (a/J)v^2 = C$$

where C is an arbitrary constant. The phase trajectories for the case $p < 0$ are shown in Fig. 3. The stationary points in this case are the centres. It can be seen that the steady motion (2.3) – rotation of the body with constant velocity about a fixed point – is realized when the moment p on the wheel balances the centrifugal force $a\omega^2$.

The case $a = 0$. When $a = 0$ Eq. (1.9) has the form

$$\begin{aligned} &v^{\cdot} - e\omega^{\cdot} + \sigma v + \gamma\sigma\omega = p \\ &(J + \theta e^2)\omega^{\cdot} + \sigma\omega + \gamma\sigma v + e\sigma v + e\gamma\sigma\omega = q + ep \end{aligned} \tag{2.5}$$

For constant values of the control parameters p and q, linear system (2.5) has a unique stationary point of the stable-node type.

$$v_0 = \frac{p - \gamma q}{\sigma(1 - \gamma^2)}, \quad \omega_0 = \frac{q - \gamma p}{\sigma(1 - \gamma^2)}$$

3. STATIONARY SOLUTIONS AND THEIR STABILITY FOR CONSTANT CONTROLS

We will now consider the general case for constant p and q. As was noted previously, without loss of generality, we will assume that $a > 0$. We will make the following changes of variables and introduce the notation

$$x = a\omega/\sigma, \quad y = av/\sigma, \quad \tau = \sigma t; \quad \tilde{p} = ap/\sigma^2, \quad \tilde{q} = aq/\sigma^2, \quad \Xi = J + \theta e^2 > 0 \tag{3.1}$$

In this notation system (1.9) takes the form (the prime denotes differentiation with respect to τ)

$$\begin{aligned} &y' - ex' - x^2 + y + \gamma x = \tilde{p} \\ &\Xi x' + xy + (1 + \gamma e)x - ex^2 + (\gamma + e)y = \tilde{q} + e\tilde{p} \end{aligned} \tag{3.2}$$

We recall that here the variable x is the normalized angular velocity of the axes connected with the body and y is the normalized linear velocity of their origin – the middle point between the wheels. The stationary points of system (3.2) are the solutions of the system of algebraic equations

$$-x^2 + y + \gamma x = \tilde{p}, \quad xy + x + \gamma y = \tilde{q} \tag{3.3}$$

which obviously reduces to the cubic equation

$$x^3 + (\tilde{p} + 1 - \gamma^2)x + \tilde{p}\gamma - \tilde{q} = 0$$

which always has at least one real root.

Suppose a certain stationary point x_0, y_0 is obtained. We will construct equations in the deviations

$$\Delta x = x - x_0, \quad \Delta y = y - y_0 \tag{3.4}$$

for system (3.2) close to this point. We substitute expressions (3.4) into system (3.2) and, taking into account Eqs (3.3), we obtain

$$\begin{aligned} &\Delta y' + \Delta y - e\Delta x' - (2x_0 - \gamma)\Delta x = \Delta x^2 \\ &\Xi\Delta x' + (1 + y_0 - 2ex_0 + \gamma e)\Delta x + (x_0 + e + \gamma)\Delta y = e\Delta x^2 - \Delta x\Delta y \end{aligned} \tag{3.5}$$

We will neglect the non-linear terms and write the characteristic equation of the linear system

$$\Xi\lambda^2 + A(x_0, y_0)\lambda + B(x_0, y_0) = 0 \tag{3.6}$$

where

$$A(x, y) = \Xi + 1 + e^2 + y - ex + 2\gamma e, \quad B(x, y) = 1 + y + 2x^2 - \gamma^2 + \gamma x \tag{3.7}$$

The straight line and parabola (3.7) isolate corresponding regions in the x, y plane. Since $\Xi > 0$, if the stationary point is in the domain $A > 0$, $B > 0$, it is stable, and if it is in the domain $B < 0$, it is a saddle.

The characteristic equation (3.6) has real roots when

$$C = A^2 - 4\Xi B \geq 0 \tag{3.8}$$

In the opposite case the roots are complex. Consequently, the curve $C(x, y) = 0$ divides the phase plane into domains, in one of which the stationary points are nodes and saddles, and in the other they are foci. Substituting Eqs (3.7) into condition (3.8) for A and B we have the following expression for this curve

$$C = (y - ex + 1 - \Xi + e^2 + 2\gamma e)^2 - 8\Xi\left(x + \frac{e+\gamma}{4}\right)^2 + \frac{9}{2}\Xi(e + \gamma)^2 = 0 \tag{3.9}$$

It is obvious that curve (3.9) is a hyperbola and that its branches are positioned outside the range of values for x determined by the inequality

$$[2x - (e + \gamma)][x + (e + \gamma)] < 0$$

It is easy to show that this hyperbola touches the parabola $B(x, y) = 0$ at the points of intersection of the latter with the straight line $A(x, y) = 0$. When $e + \gamma = 0$ hyperbola transforms into a pair of straight lines, intersecting at the point $x = 0, y = -1 + \Xi - e^2$.

Figure 4 shows the position of the curves $A = 0$, $B = 0$ (3.7) and $C = 0$ (3.9) in the x, y plane for the following values of the parameters: $\Xi = 1.0$, $\gamma = 0.2$, $e = 0.3$. The functions A and B are positive above the respective curves and the function C is positive between the branches of the hyperbola. Graphical symbols of the singular-point type in different regions of obvious meaning are also given there.

The technical difficulty of an analytical investigation of the character of stationary points of system (3.2), depending on the values of its parameters and controls p and q, as due to the fact that the coordinates of these points are calculated as the roots of a cubic equation, while coefficients (3.7) of the characteristic equation, the signs of which determine the stability, are obtained as functions of the coordinates. To avoid this difficulty we will introduce the coordinates x_0, y_0 of one stationary point; then $\tilde{p}$ and $\tilde{q}$ are determined from system (3.3), and the coordinates of the two other points, if they exist, are obtained by solving a quadric equation, after which it is possible to obtain explicit expressions A and B and analytically investigate the stability of all stationary solutions.

Suppose x_0, y_0 are the coordinates of a certain stationary point. Then the corresponding controls

$$\tilde{p}_0 = -x_0^2 + y_0 + \gamma x_0, \quad \tilde{q}_0 = x_0(y_0+1) + \gamma y_0 \tag{3.10}$$

are obtained from system (3.3), and the cubic equation, equivalent to system (3.3), obtained by substitution of (3.10), is written in the form

$$(x - x_0)(x^2 + x_0 x + y_0 + 1 - \gamma^2 + \gamma x_0) = 0 \tag{3.11}$$

If

$$G(x_0, y_0) = -x_0^2 + 4(y_0 + 1 - \gamma^2 + \gamma x_0) > 0 \tag{3.12}$$

then the stationary point is unique, otherwise there are two additional stationary points.

The curve $G = 0$ is shown in Fig. 4 for the above-mentioned values of the parameter. The function G is positive above the corresponding curve. Since $\Xi > 0$ it can be shown that the straight line $A = 0$ is always situated below the parabola $G = 0$. It is also easy to determine that the parabolas $B = 0$ and $G = 0$ always have a unique common point $x = 0, y = -1 + \gamma^2$ at which they have a common tangent.

Consequently, if a stationary point x_0, y_0 is introduced into the domain $G > 0$, then this point will be unique, $A > 0, B > 0$ for it and, hence, it is stable in the small. Additionally in this case stability occurs in the large.

We will consider the exact non-linear equations (3.5) and construct the Lyapunov function (its specific form was suggested by V. M. Morosov)

$$L = \frac{1}{2}[\Xi\Delta x^2 + (\Delta y - e\Delta x)^2] \tag{3.13}$$

the derivative of which, by virtue of Eq. (3.5), has the form

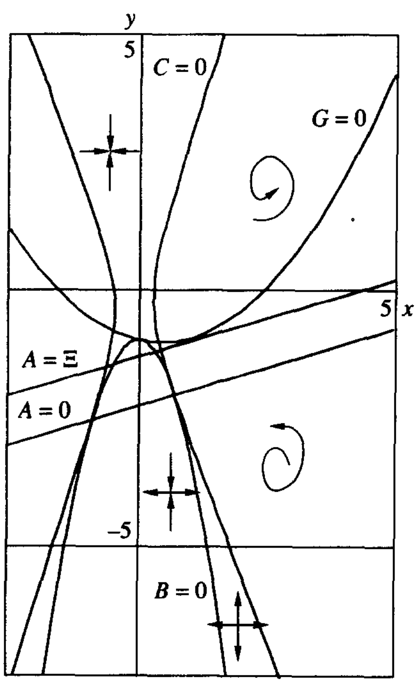


Fig. 4

$$dL/d\tau = -[(1+y_0)\Delta x^2 + (-x_0 + 2\gamma)\Delta x\Delta y + \Delta y^2] \tag{3.14}$$

The function (3.13) is positive-definite and its derivative (3.14) has the opposite sign, if the condition

$$1 + y_0 - (x_0 - 2\gamma)^2/4 > 0$$

which is identical with condition $G > 0$ (3.12) for the stationary point to be unique, is satisfied. Hence the unique stationary point is stable in the large over the whole phase plane.

Suppose now that the stationary point x_0, y_0 lies in the domain $G < 0$. Then apart from this point the system has another two stationary points. We will examine their position and stability, for which it is convenient to introduce a special parameterization. We will construct a parabola

$$y = \frac{1}{4}(1-h^2)x^2 - \gamma x - 1 + \gamma^2 \tag{3.15}$$

passing through the point $x = 0, y = -1 + \gamma^2$, common for $B = 0$, $G = 0$ and having the same tangent. It is obvious that for $h = 0$ it coincides with $G = 0$ when $h = 3 - c$ $B = 0$. Obviously, when $-\infty < x_0 < \infty$, $0 < h < 3$ we have the domain $G < 0$, $B > 0$, and when $-\infty < x_0 < \infty$, $3 < h < \infty$ we have the domain $G < 0, B < 0$. We will now specify a stationary point with parameter $h > 0$ and coordinate x_0, determining y_0 from Eq. (3.15).

The quadratic equation, produced by the multiplier in (3.11), for the coordinates $x_{1,2}$ of the other two stationary points has, in terms of these parameters, the roots

$$x_{1,2} = x_0/2(-1 \pm h) \tag{3.16}$$

The coordinates $y_{1,2}$ corresponding to them are determined from Eq. (3.10)

$$\tilde{p}_0 = -x_0^2 + y_0 + \gamma x_0 = -x_1^2 + y_1 + \gamma x_1 = -x_2^2 + y_2 + \gamma x_2 \tag{3.17}$$

and are expressed in terms of x_0 and h as follows:

$$y_{1,2} = -\frac{x_0^2}{2}(1 \pm h) + \frac{x_0}{2}\gamma(1 \mp h) - 1 + \gamma^2 \tag{3.18}$$

We will now obtain $B(x_0, y_0)$, $B(x_1, y_1)$ and $B(x_2, y_2)$, from Eq. (3.). Their expressions in terms of x_0 and h have the following simple form

$$\begin{aligned} B(x_0, y_0) &= \frac{x_0^2}{4}(9 - h^2), \quad B(x_1, y_1) = \frac{x_0^2}{2}h(-3 + h) \\ B(x_2, y_2) &= \frac{x_0^2}{2}h(3 + h) \end{aligned} \tag{3.19}$$

It is obvious, that for arbitrary x_0 and $h > 0$ one of the points is in the domain $B < 0$ and two points are outside this domain. Without loss of generality, we will further assume that $0 < h < 3$, thereby specifying the initial stationary point in the domain $G < 0$, $B > 0$. Then $B(x_0, y_0) > 0$, $B(x_1, y_1) < 0$, $B(x_2, y_2) > 0$; consequently, the stationary point x_1, y_1 is a saddle.

We will now investigate the functions $A(x_0, y_0)$, $A(x_1, y_1)$ and $A(x_2, y_2)$. From relations (3.7), (3.15), (3.16) and (3.18) we can obtain

$$\begin{aligned} A(x_0, y_0) &= \left(\varepsilon - \frac{1-h}{2}x_0\right)\left(\varepsilon - \frac{1+h}{2}x_0\right) + \Xi \\ A(x_1, y_1) &= (\varepsilon + x_0)\left(\varepsilon - \frac{1+h}{2}x_0\right) + \Xi \\ A(x_2, y_2) &= (\varepsilon + x_0)\left(\varepsilon - \frac{1-h}{2}x_0\right) + \Xi; \quad \varepsilon = \gamma + e \end{aligned} \tag{3.20}$$

Considering the signs of the factors in the expressions for $A(x_0, y_0)$ and $A(x_2, y_2)$ (it is not necessary to investigate $A(x_1, y_1)$) it can be established that for any x_0, ε and $0 < h < 3$ one of these products will be positive and the other negative. Since $\Xi > 0$, then either both functions $A(x_0, y_0)$ and $A(x_2, y_2)$ are positive or one of them is. Consequently, either both stationary points (x_0, y_0) and (x_2, y_2) are stable or one of them is.

We will consider separately the case when the point x_0, y_0 lies on the straight line $A = \Xi$. It can be seen from Eqs (3.20) that one other point x_2, y_2 for $x_0 > 2\varepsilon$ or x_1, y_1 for $x_0 < 2\varepsilon$ lies on the same straight line. Here we assume that $\varepsilon = e + \gamma > 0$, which corresponds to Fig. 4. On the other hand, in the case considered $y_0 = ex_0 - 1 - e^2 - 2\gamma e$, in accordance with expressions (3.7). Taking this relation into account we can write expressions (3.10) for $\tilde{p}$, $\tilde{q}$ and determine that the straight line $x = -\varepsilon$ is the solution of the dynamic equations (3.2) and that a third singular point is positioned on it. For $x_0 < 2\varepsilon$ the straight line $x = -\varepsilon$ passes through the stable singular point (node) x_2, y_2, and the saddle point x_1, y_1 lies on the straight line $A = \Xi$. Both points merge for $x_0 = 2\varepsilon$. For $x_0 > 2\varepsilon$ the stable point x_2, y_2 lies on the straight line $A = \Xi$ and the saddle point x_1, y_1 belongs to the straight line $x = -\varepsilon$, which is a vertical separatrix dividing the domains of attraction of two stable singular points. Note that in the symmetrical case $e = \gamma = 0$ only the last version is realized.

Thus, the non-holonomic mechanical system considered can have either one or three stationary points; the unique stationary point is stable, and stable in the large; if there are three stationary points, then one of them is a saddle point and the other two are nodes or foci, and at least one of them is stable.

4. PHASE TRAJECTORIES

The phase portrait of the system is also generally clear after the number, position and character of the stationary points of the system have been investigated. We will cite here (Fig. 5) typical positions of phase trajectories for the same parameter values $\Xi = 1.0$, $\gamma = 0.2$, $e = 0.3$ as previously for the change in position of the specified stationary point. Thus, the qualitative characteristics to not depend on the choice of the particular parameters of the mechanical system. The choice of the stationary point in the half-planes $y > 0$ corresponds to positive linear velocities, when the centre of gravity is shifted forward relative to the axis of the wheels in the direction of motion. The characteristic lines are shown by the dashed line in Fig. 4.

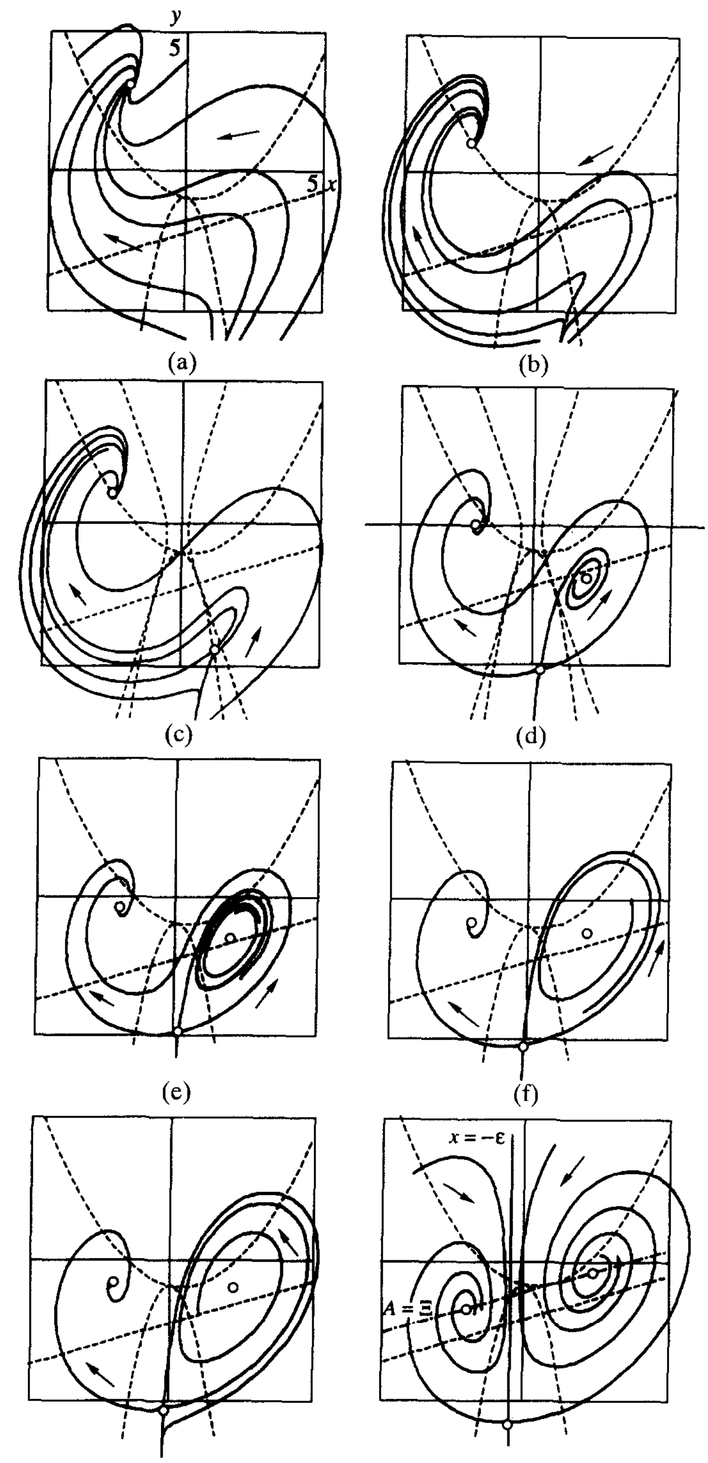
In Fig. 5(a) the stationary point is chosen in the domain $G > 0$ and, hence, is unique and stable in the large. Changes in the phase plane occur when this point is shifted downwards. These are shown in Fig. 5(b–h). A deformation of the phase trajectories occurs (Fig. 5b) on approaching the boundary $G = 0$ of the domain of uniqueness of the stationary point, and if the stationary point falls on the boundary $G = 0$ (Fig. 5c) a new multiple stationary point develops on the curve $B = 0$. Furthermore, this multiple singular point splits into two, one of them remains a saddle points and the second is initially an unstable node and then an unstable focus (Fig. 5d).

When the left focus moves further downward, the right, unstable focus shifts upwards, and on intersecting the straight line $A = 0$ transforms into a centre and further on into a stable focus. In addition, an unstable limiting cycle (Fig. 5e) appears around it. The limiting cycle increases in size and at a certain instant the two separatrices of the saddle point are closed in it (Fig. 5f), after which it disappears and the domain of attraction of the right focus becomes unbounded (Fig. 5g) along the lower separatrix of the saddle. Further, when the left focus moves downward, its domain of attraction decreases due to the right focus and when it falls on the straight line $A = \Xi$ the second focus also appears on this straight line. In this case, the domains of attraction are half-planes, divided by the vertical straight line $x = -\varepsilon = -(e, \gamma)$, on which the saddle point is situated (Fig. 5h).

The further changes occur in reverse order: the domain of attraction of the left focus decreases, around it an unstable limiting cycle develops, etc.

The evolution described applies to the case when the multiple singular point develops in the domain $A < 0$ and breaks down into an unstable node and a saddle. If the multiple point appears in the domain $A > 0$, it breaks down into a stable node and a saddle, and the domain of attraction of the node immediately becomes unbounded.

Note the following feature of the system. In the case of two stable stationary movies (Fig. 5e–g), they are realized for the same control and a change from one mode to the other by a single change of control becomes impossible.



(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 5

5. CONCLUSION

1. If the vector of the given linear velocity of the robot is in the same direction as the shift in the centre of gravity of the robot relative to the axis of the drive wheels, then for the appropriate constant controls the motion of the robot is stable. In the opposite case the stability of the specified motion is conserved only for considerable limits on the magnitude of the linear and angular velocities. A breach of these

limits leads to the specified motion being unstable and the robot reaches a stable type of motion different from the specified one. Of course if the wheels of the robot are equipped with sensors of their singular velocities, then, using the readings of these sensors, it is possible to form the control signal, stabilizing any motion, but it is obvious that the costs of stabilizing naturally stable motion are considerably less than for unstable motion.

2. Apart from the multiplicity of stationary solutions for constant controls in the system, the existence of singular solutions, such as unstable limiting cycles, is possible.

3. Using the relations and graphs obtained it is easy to interpret the influence of asymmetry related to the position of the centre of gravity and the different parameters of the motors. But it should be noted that asymmetry does not lead to any qualitative change in the behaviour of the system.

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REFERENCES

1. CHAPLYGIN, S. A. The theory of non-holonomic systems, a theorem on the reducing factor. *Mat. Sbornik*, 1911, **28**, 2, 303–314.
2. APPELL, P., Exemple du mouvement d'un systeme assujetti à une liason exprimée par une relation non linéaires entre des composantes de la vitesse. *Rend. circ. mat. di Palermo*, 1911, **32**, 1–3.
3. CARATHEODORY, C., Der Schlitten. *ZAMM*, 1993, **13**, 2, 71–76.
4. HAMEL., G., *Theoretische Mechanik*. Berlin, Springer, 1949.

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